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Inequalities for Quasi-Symmetric Designs

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A 2-design is said to be quasi-symmetric if there are two block intersection sizes. We obtain inequalities satisfied by the parameters of a quasi-symmetric design using linear programming techniques. The same methods apply to codes with covering radius 2 with the property that the number of codewords at distance 2 from a given vector v depends on the distance of v from the code. © 1988 Academic Press, Inc.

1. INTRODUCTION

An *association scheme with n* classes on a set X is a partition of the 2-element subsets of X into n classes $R_1, ..., R_n$ satisfying

(1) given $x \in X$ the number v_i of $y \in X$ with $\{x, y\} \in R_i$ depends only on i;

(2) given $x, y \in X$ with $\{x, y\} \in R_k$, the number of $z \in X$ with $\{x, z\} \in R_i$ and $\{y, z\} \in R_j$ is a constant p_{ij}^k depending only on i, j and k.

The Johnson scheme $J(v, k)$ is an association scheme with k classes. The point set X is the set of all k-subsets of a v-set and $\{x, y\} \in R_i$ if and only if $|x \cap y| = k - i$. The Hamming scheme $H(v, q)$ is an association scheme with v classes. The point set $X = \mathrm{GF}(q)^v$ and a pair of vectors $\{x, y\} \in R_i$ if and only if the Hamming distance $d(x, y) = i$.

Let $D_0 = I$ and let D_i be the adjacency matrix of the graph (X, R_i). The matrices $D_0, D_1, ..., D_n$ are symmetric and they span a commutative $n+1$ dimensional real algebra called the Bose-Mesner algebra of the scheme. Since the Bose-Mesner algebra is semisimple it admits a unique basis of mutually orthogonal idempotent matrices $J_0, J_1, ..., J_n$. Writing $D_l = \sum_{i=0}^{n} P_l(i) J_i$, for $l = 0, 1, ..., n$, we have $D_l J_i = P_l(i) J_i$. Thus $P_l(i)$ is the eigenvalue of D_l associated with the eigenspace V_i spanned by the columns of J_i. The $(n+1) \times (n+1)$ matrix P with ilth entry $p_l(i)$ is called the *eigenmatrix* of

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the scheme. The matrix $Q=|X|\,P^{-1}$ with ilth entry $q_l(i)$ is called the *dual eigenmatrix*. Note that

$$J_l=|X|^{-1}\sum_{i=0}^{n} q_l(i)\,D_i. \tag{1}$$

Let $\mathscr{F}$ be a non-empty subset of the point set X. The *inner distribution* $a=(a_0, ..., a_n)$ of $\mathscr{F}$ is given by

$$a_i=\frac{1}{|\mathscr{F}|}\sum_{x,\,y\in\mathscr{F}} D_i(x, y),$$

which is the average valency of R_i restricted to $\mathscr{F}$. The *degree* $s(\mathscr{F})$ is the number of non-zero components of the inner distribution not counting $a_0=1$. In the Johnson scheme $J(v, k)$, the elements of $\mathscr{F}$ are called *blocks* and the *degree* $s(\mathscr{F})$ is the number of possible block intersection sizes. In the Hamming scheme $H(v, q)$, the degree $s(\mathscr{F})$ is the number of possible distances between distinct vectors in $\mathscr{F}$.

Let π be the characteristic vector of $\mathscr{F}$ and let π_i be the projection of π onto the eigenspace V_i. The dual distribution $b=(b_0, b_1, ..., b_n)$ is defined by

$$b_i=\frac{|X|}{|\mathscr{F}|^2}\,\pi^T J_i \pi=\frac{|X|}{|\mathscr{F}|^2}\,\|\pi_i\|^2. \tag{3}$$

From (1), (2), and (3) it follows that

$$\frac{1}{|\mathscr{F}|}\,aQ=b. \tag{4}$$

Then $b_0=1$ and $b_i\geqslant 0$ for $1\leqslant i\leqslant n$. The subset $\mathscr{F}$ is said to have strength t if $b_1=b_2=\cdots=b_t=0$. In the Johnson scheme $J(v, k)$, a subset $\mathscr{F}$ with strength t is a t-design. On the other hand, if $\mathscr{F}$ is a linear code in the Hamming scheme $H(v, q)$, then a_i is the number of codewords of weight i in $\mathscr{F}$, and b_i is the number of codewords of weight i in $\mathscr{F}^{\perp}$. We shall apply the non-negativity of b_3, b_4 to restrict the parameters of subsets $\mathscr{F}$ with degree 2 and strength 2.

A 2-design $\mathscr{F}$ is said to be quasi-symmetric if there are two block intersection sizes, say $k-x$ and $k-y$. Quasi-symmetric 2-designs are precisely those subsets of $J(v, k)$ with degree 2 and strength 2. One interesting property of quasi-symmetric designs is that the block graph is strongly regular (Goethals and Seidel [6]). Define

$$f_3(x, y, k, v) = (v-1)(v-2)\,xy - k(v-k)(v-2)(x+y) + k(v-k)(k(v-k)-1), \tag{5}$$

$$\begin{aligned} f_4(x, y, k, v) = & -(v-6)(v-3)(v-1)\,xy(x+y) \\ & + (v-6)(v-3)\,k(v-k)(x+y)^2 \\ & - 2(v-3)\,k(v-k)(2k(v-k)-3v)(x+y) \\ & + (v-3)(k(v-k)(3v+2) - 6v(v-1))\,xy \\ & + k(v-k)(3k(v-k)(k(v-k)-2(v-1)) + 5v - 3). \end{aligned} \tag{6}$$

The main result of Section 2 is the following theorem.

THEOREM. *Let $\mathscr{F}$ be a $2-(v, k, \lambda)$ design such that $|p \cap q| = k - x$ or $k - y$ for all pairs of distinct blocks $p, q \in \mathscr{F}$. Then*

(i) $f_3(x, y, k, v) \geqslant 0$,

(ii) $f_4(x, y, k, v) \geqslant 0$.

Equality holds in (i) *if and only if $\mathscr{F}$ is a* 3*-design. If $\mathscr{F}$ is a* 3*-design then*

(iii) $x + y \geqslant 2k - 1 - 2(k-1)(k-2)/(v-3)$,

and equality holds in (iii) *if and only if $\mathscr{F}$ is a* 4*-design.*

Neumaier [8; 9, Proposition 12] derived an inequality satisfied by the parameters of a quasi-symmetric 2-design that is equivalent to $f_3(x, y, k, v) \geqslant 0$. However it is not as simple to state. The inequality $f_4(x, y, k, v) \geqslant 0$ eliminates quasi-symmetric 2-(28, 7, 16) and 2-(29, 7, 12) designs. Tonchev [12] proved that these designs do not exist using the classification of self-dual binary [30, 15, 6] codes. Quasi-symmetric 2-designs that are also 4-designs have been classified by Ito [7] and Bremner [2]. The unique 4-(23, 7, 1) design and its complement are the only examples.

In Section 3 we apply the same methods to codes $\mathscr{F}$ in the Hamming scheme $H(v, q)$ with degree 2 ($b_0 = 1$, $b_x, b_y \neq 0$, $b_i = 0$ for $i \neq x, y$) and minimum distance $d \geqslant 3$. We define polynomials

$$h_3(x, y, q, v) = (qx - v(q-1) - 1)(qy - v(q-1) - 1) + (v-1)(q-1), \tag{7}$$

$$\begin{aligned} h_4(x, y, q, v) = & -q^3xy(x+y) + (v(q-1)+1)\,q^2(x+y)^2 \\ & + q^2(3nq - 6q - 3n + 11)\,xy - 2(v(q-1)+1)(2v(q-1) \\ & - 3(q-2))q(x+y) + 3(v-2)v^2q^3 - (9v^3 - 30v^2 + 18v - 5)q^2 \\ & + 3(v-3)(v-1)(3v-2)\,q - 3(v-3)(v-2)(v-1), \end{aligned} \tag{8}$$

and we prove the following result.

THEOREM. *Let $\mathscr{F}$ be a code in $H(v, q)$ with degree 2 and minimum distance $d \geqslant 3$. Then*

(i) $h_3(x, y, q, v) \geqslant 0$

(ii) $h_4(x, y, q, v) \geqslant 0$.

Equality holds in (i) *if and only if* $d \geqslant 4$. *If* $d \geqslant 4$ *then*

(iii) $q(x+y+1) \geqslant 2v(q-1)+4$,

and equality holds in (iii) *if and only if* $\mathscr{F}$ *is a perfect 2-error-correcting code.*

2. Quasi-Symmetric Block Designs

In this section we consider a family $\mathscr{F}$ of k-subsets of a v-set such that every pair of blocks meet in $k-x$ points or $k-y$ points. We use the framework of the Johnson scheme $J(v, k)$ to relate the intersection properties of $\mathscr{F}$ to the covering properties of $\mathscr{F}$. For the Johnson scheme $J(v, k)$ the eigenvalue $P_l(i) = E_l(i)$ where $E_l(x)$ is an Eberlein (dual Hahn) polynomial defined by

$$E_l(x) = \sum_{j=0}^{l} (-1)^j \binom{x}{j} \binom{v-x}{l-j} \binom{v-k-x}{l-j}, \qquad l = 0.\ 1, ..., k.$$

See Delsarte [3] or [5] for details. The ilth entry of the dual eigenmatrix Q is $q_l(i) = H_l(i)$, where $H_l(z)$ is a Hahn polynomial defined by

$$H_l(z) = \left[\binom{v}{l} - \binom{v}{l-1}\right] \sum_{i=0}^{l} \left\{(-1)^i \binom{l}{i} \binom{v+1-l}{i} \binom{k}{i}^{-1} \binom{v-k}{i}^{-1}\right\} \binom{z}{i},$$

$$l = 0, 1, ..., k.$$

The Hahn polynomials satisfy the recurrence relation

$$\gamma_{l+1} H_{l+1}(z) = (\alpha_l - z)\, H_l(z) - w_l \gamma_l H_{l-1}(z),$$

where

$$\gamma_l = \frac{l(v-k-l+1)(k-l+1)}{(v-2l+1)(v-2l+2)},$$

$$\alpha_l = \frac{(v-k)\, k(v+2) - vl(v-l+1)}{(v-2l)(v-2l+2)},$$

$$w_l = \frac{(v-2l+1)(v-l+2)}{l(v-2l+3)}.$$

The first three Hahn polynomials are

$$H_0(z) = 1$$

$$H_1(z) = \frac{(v-1)}{k(v-k)}(k(v-k) - vz), \tag{9}$$

$$H_2(z) = \frac{v(v-3)}{2k(k-1)(v-k)(v-k-1)}(z^2(v-1)(v-2) - z(v-1)(2k(v-k)-v) + k(k-1)(v-k)(v-k-1)).$$

To begin, we assume only that $\mathscr{F}$ is a 1-design. The first theorem is joint work with Lenore Cowen.

THEOREM 1. *Let $\mathscr{F}$ be a family of k-subsets of a v-set such that $\mathscr{F}$ is a 1-design and such that $|p \cap q| = k-x$ or $k-y$ for all pairs of distinct blocks p, q in $\mathscr{F}$. Then*

$$\frac{|\mathscr{F}|-1}{|\mathscr{F}|} \leqslant \frac{k(v-k)}{v(v-1)}\left[\frac{(v-1)(x+y)-k(v-k)}{xy}\right], \tag{10}$$

and equality holds if and only if $\mathscr{F}$ is a 2-design.

Proof. Since $\mathscr{F}$ is a 1-design

$$|\mathscr{F}|\, b_1 = 0 = H_1(0) + a_x H_1(x) + a_y H_1(y).$$

If we define

$$G_1(z) = \frac{k(v-k)}{(v-1)} H_1(z) = k(v-k) - vz,$$

then

$$(1, a_x, a_y) = |\mathscr{F}|\left(0, \frac{-G_1(y)}{G_1(x)-G_1(y)}, \frac{G_1(x)}{G_1(x)-G_1(y)}\right) + \left(1, \frac{y}{x-y}, \frac{x}{y-x}\right). \tag{11}$$

Now

$$|\mathscr{F}|\, b_2 = \sum_{i=0}^{k} a_i H_2(i) = H_2(0)\left[\sum_{i=0}^{k} a_i\left(\frac{H_2(i)}{H_2(0)} - \frac{H_1(i)}{H_1(0)}\right)\right] \geqslant 0, \tag{12}$$

and by direct calculation

$$\frac{H_2(z)}{H_2(0)} - \frac{H_1(z)}{H_1(0)} = -\frac{(v-2)}{k(v-k)} z(k(v-k) - z(v-1)).$$

If we define

$$G_2(z)=z(k(v-k)-(v-1)z)$$

then (12) becomes

$$(1, a_x, a_y)(0, G_2(x), G_2(y))^T \leqslant 0. \tag{13}$$

The inequality (10) follows directly from (11) and (13). Equality holds if and only if $b_2=0$. That is equality holds if and only if $\mathscr{F}$ is a 2-design.

Remarks. (1) Let $\mathscr{F}$ be a quasi-symmetric 2-design with block intersection sizes $k-x$ and $k-y$. Theorem 1 implies that if $\mathscr{B}$ is a 1-design with block intersection sizes $k-x$ and $k-y$, then $|\mathscr{B}| \leqslant |\mathscr{F}|$, with equality if and only if $\mathscr{B}$ is a 2-design.

(2) If $G_1(x)<0$ then the inequality

$$a_y=|\mathscr{F}|\left(\frac{k(v-k)-vx}{v(y-x)}\right)+\frac{y}{y-x}>0$$

gives the bound

$$\frac{|\mathscr{F}|-1}{|\mathscr{F}|}<\frac{k(v-k)}{vx}. \tag{14}$$

Let $\mathscr{F}$ be a family of 4-subsets of a 12-set such that $\mathscr{F}$ is a 1-design and $|p\cap q|=0$ or 1 for all pairs of distinct blocks p, q. Then $|\mathscr{F}| \equiv 0 \pmod 3$, and it is easy to construct a family $\mathscr{F}$ of size 6. The bound provided by Theorem 1 is $|\mathscr{F}| \leqslant 11$ whereas the bound provided by (14) is $|\mathscr{F}|<9$. The family with 6 blocks is extremal.

(3) Let

$$s(x, y)=\frac{k(v-k)}{v(v-1)}\left[\frac{(v-1)(x+y)-k(v-k)}{xy}\right].$$

Then

$$\frac{\partial s}{\partial x}=\frac{1}{x^2y}G_1(y) \qquad \text{and} \qquad \frac{\partial s}{\partial y}=\frac{1}{y^2x}G_1(x).$$

Given the bound (14), it is natural to suppose $G_1(y)<0$ and $G_1(x)>0$. Then $s(x, y)$ is a decreasing function of x and an increasing function of y. There are values x, y, k, v for which $s(x, y) \geqslant 1$ and Theorem 1 provides no bound on $|\mathscr{F}|$. For example, if $x=1$, $y=3$, $k=5$, $v=8$ then $s(x, y)=\frac{65}{56}$.

(4) Suppose $y > 2x$. Then the relation $p \sim q$ if $|p \cap q| = k - x$ is an equivalence relation on blocks and a_x is the size of every equivalence class. Let A be the incidence matrix of the family $\mathscr{F}$. Then the eigenvalues of AA^T are $|\mathscr{F}|\, k^2/v$ with multiplicity 1, $(a_x - 1)(y - x)$ with multiplicity $|\mathscr{F}|/a_x - 1$, and x with multiplicity $|\mathscr{F}|(a_x - 1)/a_x$. Thus A is non-singular and $|\mathscr{F}| \leqslant v$. Examples of extremal families can be constructed from symmetric designs as follows. Let S, T be disjoint $v/2$-sets, let x_i, $i = 1, 2, ..., v/2$, be the blocks of a symmetric $(v/2, l, \lambda)$ design on S, and let y_i, $i = 1, 2, ..., v/2$, be the blocks of a symmetric $(v/2, l, \lambda)$ design on T. The sets $S \cup y_i$, $T \cup x_i$, $i = 1, 2, ..., v/2$, define an extremal family $\mathscr{F}$ with $k = v/2 + l$, $y = v/2 - l$, $x = l - \lambda$. Extremal families constructed in this way are examples of group divisible designs (see Bose and Connor [1]).

The main result of this section is the following theorem (the polynomials $f_3(x, y, k, v)$ and $f_4(x, y, k, v)$ are defined in (5) and (6)).

THEOREM 2. *Let $\mathscr{F}$ be a $2-(v, k, \lambda)$ design such that $|p \cap q| = k - x$ or $k - y$ for all pairs of distinct blocks $p, q \in \mathscr{F}$. Then*

(i) $f_3(x, y, k, v) \geqslant 0$,

(ii) $f_4(x, y, k, v) \geqslant 0$.

Equality holds in (i) *if and only if $\mathscr{F}$ is a 3-design. If $\mathscr{F}$ is a 3-design then*

$$\text{(iii)} \quad x + y \geqslant 2k - 1 - \frac{2(k-1)(k-2)}{(v-3)},$$

and equality holds in (iii) *if and only if $\mathscr{F}$ is a 4-design.*

Proof. Since $\mathscr{F}$ is a 2-design it follows from (13) that

$$(1, a_x, a_y)(0, G_2(x), G_2(y))^T = 0,$$

and so

$$(a_x, a_y) = c(-G_2(y), G_2(x)), \tag{15}$$

for some positive constant c. Now

$$|\mathscr{F}|\, b_3 = \sum_{i=0}^{k} a_i H_3(i) = H_3(0) \left[\sum_{i=0}^{k} a_i \left(\frac{H_3(i)}{H_3(0)} - \frac{H_2(i)}{H_2(0)} \right) \right] \geqslant 0. \tag{16}$$

The three-term recurrence relation satisfied by the Hahn polynomials gives

$$\begin{aligned} H_2(i)\, H_3(0) - H_2(0)\, H_3(i) &= H_2(i)\left(\frac{1}{\gamma_3}(\alpha_2 H_2(0) - w_2\gamma_2 H_1(0))\right) \\ &\quad - H_2(0)\left(\frac{1}{\gamma_3}((\alpha_2 - i)\, H_2(i) - w_2\gamma_2 H_1(i))\right) \\ &= \frac{1}{\gamma_3}(iH_2(0)\, H_2(i) + w_2\gamma_2(H_1(i)\, H_2(0) - H_1(0)\, H_2(i))). \end{aligned}$$

Since $\mathscr{F}$ is a 2-design

$$\sum_{i=0}^{k} a_i(H_1(i)\, H_2(0) - H_1(0)\, H_2(i)) = 0$$

and (16) becomes

$$(a_x, a_y)(-xH_2(x), -yH_2(y))^T \geqslant 0.$$

This condition simplifies to give

$$\begin{aligned} &(v-1)(y-x)\, yx[(v-1)(v-2)\, xy - k(v-k)(v-2)(x+y) \\ &\quad + k(v-k)(k(v-k)-1)] \geqslant 0, \end{aligned}$$

which completes the proof of (i).

Now

$$|\mathscr{F}|\, b_4 = \sum_{i=0}^{k} a_i H_4(i) = H_4(0)\left[\sum_{i=0}^{k} a_i\left(\frac{H_4(i)}{H_4(0)} - \frac{H_1(i)}{H_1(0)}\right)\right] \geqslant 0. \tag{17}$$

Write

$$\frac{H_4(z)}{H_4(0)} - \frac{H_1(z)}{H_1(0)} = zG_3(z)$$

and rewrite (17) as

$$(-G_2(y), G_2(x))(xG_3(x), yG_3(y))^T \geqslant 0. \tag{18}$$

Part (ii) follows from (18) after some routine but tedious calculations.

Equality holds in (i) if and only if $\mathscr{F}$ is a 3-design. If $f_3(x, y, k, v)=0$ then

$$f_4(x, y, k, v) = -\frac{(v-6)\,k(k-1)(v-k)(v-k-1)}{(v-2)(v-1)} \times (2k(v-k)-v-1-(x+y)(v-3))$$

and (iii) holds. Equality holds in (iii) if and only if $\mathscr{F}$ is a 4-design.

Remarks. (1) Quasi-symmetric 2-designs that are 4-designs are called tight 4-designs (see Ray-Chaudhuri and Wilson [10]). The integers x, y are the roots of the quadratic equation

$$z^2 - \left[2k - 1 - \frac{2(k-2)(k-1)}{(v-3)}\right] z + \frac{k(v-k)}{(v-2)(v-3)}(k(v-k)-(v-1)) = 0.$$

This observation was the starting point for the classification of tight 4-designs by Ito [7] and Bremner [2].

(2) Consider an arbitrary association scheme. For any choice of $n+1$ distinct real numbers $z_0 = 0, z_1, ..., z_n$ we can find polynomials $\Phi_0, \Phi_1, ..., \Phi_n$ each of degree at most n such that $\Phi_k(z_i) = q_k(i)$ for $0 \leqslant i, k \leqslant n$. The association scheme is said to be *Q-polynomial* if there is a choice of $z_0 = 0, z_1, ..., z_n$ for which Φ_k has degree k for $0 \leqslant k \leqslant n$. In a Q-polynomial scheme the polynomials Φ_l satisfy a 3-term recurrence relation

$$\gamma_{l+1}\Phi_{l+1}(z) = (\alpha_l - z)\,\Phi_l(z) - w_l\gamma_l\Phi_{l-1}(z),$$

where α_l, w_l, γ_l are constants. Let $\mathscr{F}$ be a subset with degree 2 ($a_0 = 1, a_x, a_y \neq 0$, and $a_i = 0$ for $i \neq 0, x, y$) and strength 2. Then the method used to prove part (i) of Theorem 2 gives

$$\alpha_0(\gamma_1^2 w_1 - \alpha_0\alpha_1)\,xy - (\gamma_1^2 w_1 + \alpha_0^2)(\gamma_1^2 w_1 - \alpha_0\alpha_1)(x+y) + (\gamma_1^2 w_1 - \alpha_0\alpha_1)(\alpha_1\gamma_1^2 w_1 + 2\alpha_0\gamma_1^2 w_1 + \alpha_0^3) \geqslant 0$$

with equality if and only if $\mathscr{F}$ has strength 3.

(3) The condition $f_4(x, y, k, v) \geqslant 0$ eliminates quasi-symmetric $2-(28, 7, 16)$ designs ($v=28$, $k=7$, $x=4$, $y=6$) since $f_4(4, 6, 7, 28) = -1350$. It eliminates quasi-symmetric $2-(29, 7, 12)$ designs ($v=29$, $k=7$, $x=4$, $y=6$) since $f_4(4, 6, 7, 29) = -252$.

3. The Hamming Scheme $H(v, q)$

In this section we consider subsets $\mathscr{F}$ of $H(v, q)$ with degree 2 and strength at least 2. These subsets $\mathscr{F}$ include uniformly packed single-error-correcting codes. They are a generalization of perfect codes and they were introduced by Semakov, Zinovjev, and Zaitzev in [11]. The spheres of radius 2 about the codewords cover the whole space and these spheres overlap in a very regular way. There are constants λ and μ (with $\lambda < (n-e)(q-1)/(e+1)$) such that vectors at distance 1 from the code are in $\lambda+1$ spheres and vectors at distance 2 from the code are in μ spheres. If the restriction on λ were removed, a perfect code would also be uniformly packed.

For the Hamming scheme $H(v, q)$

$$P_l(i) = q_l(i) = K_l(i),$$

where

$$K_l(z) = \sum_{i=0}^{l} (-1)^i (q-1)^{l-i} \binom{z}{i} \binom{v-z}{l-i}$$

is the lth Krawtchouk polynomial.

The main result of this section is the following theorem (the polynomials $h_3(x, y, q, v)$ and $h_4(x, y, q, v)$ are defined in (7) and (8)). Since the theorem is proved by the same method as Theorem 2, we just sketch the details.

THEOREM 3. *Let $\mathscr{F}$ be a code in $H(v, q)$ with degree 2 ($b_0 = 1$, $b_x, b_y \neq 0$, $b_i = 0$ for $i \neq 0, x, y$) and minimum distance $d \geqslant 3$. Then*

(i) $h_3(x, y, q, v) \geqslant 0$,

(ii) $h_4(x, y, q, v) \geqslant 0$.

Equality holds in (i) *if and only if $d \geqslant 4$. If $d \geqslant 4$ then*

(iii) $q(x+y+1) \geqslant 2v(q-1)+4$,

and equality holds in (iii) *if and only if $\mathscr{F}$ is a perfect 2-error-correcting code.*

Proof. Since the minimum distance $d > 2$,

$$\frac{q^v}{|\mathscr{F}|} a_2 = 0 = \sum_{i=0}^{v} b_i \left[\frac{K_2(i)}{K_2(0)} - \frac{K_1(i)}{K_1(0)} \right]$$

Direct calculation gives

$$\frac{K_2(z)}{K_2(0)}-\frac{K_1(z)}{K_1(0)}=\frac{qz(qz-n(q-1)-1)}{n(n-1)(q-1)^2}$$

so that if

$$M_2(z)=qz(qz-n(q-1)-1),$$

then

$$(b_x, b_y)=C(-M_2(y), M_2(x)) \tag{19}$$

for some positive constant C. The condition

$$\frac{q^v}{|\mathscr{F}|}a_3=K_3(0)\sum_{i=0}^{v} b_i\left[\frac{K_3(i)}{K_3(0)}-\frac{K_2(i)}{K_2(0)}\right]\geqslant 0$$

can be rewritten in the form

$$(b_x, b_y)(-xK_2(x), -yK_2(y))^T\geqslant 0. \tag{20}$$

Using (19) to substitute for (b_x, b_y) we obtain

$$(qx-v(q-1)-1)(qy-v(q-1)-1)+(v-1)(q-1)\geqslant 0. \tag{21}$$

Equality holds in (21) if and only if $a_3=0$ and $d\geqslant 4$.

Just as in Theorem 2, the inequality

$$\frac{q^v}{|\mathscr{F}|}b_4=H_4(0)\sum_{i=0}^{v} b_i\left[\frac{K_4(i)}{K_4(0)}-\frac{K_1(i)}{K_1(0)}\right]\geqslant 0$$

simplifies to give $h_4(x, y, q, v)\geqslant 0$ as required. If $h_3(x, y, q, v)=0$ then

$$h_4(x, y, q, v)=-(v-1)(q-1)(2q(v-1)+4-q(x+y+1))$$

and (iii) holds. Equality holds in (iii) if and only if $\mathscr{F}$ is a perfect 2-error-correcting code.

Note added in proof. Since this paper was accepted for publication the author has learned that W. H. Haemers was the first to eliminate quasi-symmetric $2-(28, 7, 16)$ and $2-(29, 7, 12)$ designs (W. H. Haemers, "Sterke Grafen en Block Designs," Masters thesis, Eindhoven Univ. of Technology, 1975 (in Dutch)). A. E. Brouwer and E. J. L. J. van Heyst have found that for $k<1000$ the only solutions to $f_3(x, y, k, v)=0$ correspond to extensions of symmetric designs or to designs associated with the Witt designs. Sane and Shrikhande have conjectured that these are the only examples (S. S. Sane and M. S. Shrikhande, Quasi-symmetric 2, 3, 4-designs, *Combinatorica*, to appear).

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